

Noncommutative deformations of quantum field theories, locality, and causality

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Abstract

We investigate noncommutative deformations of quantum field theories for different star products, particularly emphasizing the locality properties and the regularity of the deformed fields. Using functional analysis methods, we describe the basic structural features of the vacuum expectation values of star-modified products of fields and field commutators. As an alternative to microcausality, we introduce a notion of θ -locality, where θ is the noncommutativity parameter. We also analyze the conditions for the convergence and continuity of star products and define the function algebra that is most suitable for the Moyal and Wick-Voros products. This algebra corresponds to the concept of strict deformation quantization and is a useful tool for constructing quantum field theories on a noncommutative space-time.

Key words: noncommutative quantum field theory, Moyal and Wick-Voros star products, locality, causality, topological algebras of entire functions

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I. Introduction

The recent intensive study of noncommutative field theory models have been motivated by fundamental issues in quantum physics and gravity [1]. Although the idea of space-time noncommutativity is old and, as well known, goes back to Heisenberg and Snyder, the development of string theory has inspired a renewed interest in this research field (see, e.g., [2] for a review). Almost all works on this subject begin by replacing the ordinary pointwise product of functions on space-time with a noncommutative star product. As a result, the coordinate functions satisfy the commutation relation

$$[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \quad (1)$$

where $\theta^{\mu\nu}$ is a real antisymmetric matrix playing the role of a noncommutativity parameter and determining a deformation of the Minkowski space. But this deformation can be combined with the basic principles of quantum theory many ways. Moreover, there are many star products compatible with commutation relation [1]. The great majority of authors use the Moyal-Weyl-Grönewold star product, but a field theory with the Wick-Voros (or normally ordered) star product was recently discussed [3, 4]. Interestingly, the authors reached opposite conclusions about the equivalence of theories with different star products.

There is not yet a consensus about the best way of constructing a self-consistent noncommutative field theory. The so-called twisted Poincaré covariance, aimed at restoring the space-time symmetries broken by noncommutativity, is much considered (see [5, 6] and the references therein). How to implement this covariance is also debated, but the notion of a twisted tensor product associated with a given star product plays a major role in all implementations. Since we lack a generally accepted formulation of noncommutative quantum field theory, the question of the causality of observables merits further investigation, because it is crucial for the physical interpretation.

We note that the star products and twisted tensor products can be treated from two different standpoints. It is more common to regard them as formal series in powers of the deformation parameter θ . But this approach is not quite acceptable physically. When analyzing the causal structure of noncommutative models, we should take into account that these products are inherently nonlocal because they are determined by infinite-order differential operators. To gain a better insight into these nonlocal features, we must establish the conditions under which the formal series converge and describe the corresponding topological algebras in which the star product depends continuously on the deformation parameter. In what follows, we adhere to the second approach, which is close to the strategy of strict deformation quantization [7].

The plan of presentation is as follows. We first recall the definitions of the Moyal and Wick-Voros star products and their associated tensor products. We then show that the Moyal twisted tensor product can be used to naturally deform [8, 9] a quantum field theory initially defined on commutative Minkowski space and describe basic properties of the resulting noncommutative field theory. In particular, the deformed quantum fields can be localized in certain wedge-shaped regions [9], but this localization differs essentially from the modification of the light cone to a light wedge that was previously proposed [10, 11, 12] for theories with space-space noncommutativity describing the low-energy limit of string theory. Further, we consider the convergence conditions for the formal power series determining star products and

describe the basic properties of the corresponding function algebras [13]. This analysis and some instructive examples lead to the concept of θ -locality [14] which in our opinion can be used to implement causality in a noncommutative quantum field theory.

II. The Moyal and Wick-Voros star products

The formal power series representing the Moyal and Wick-Voros star products are written as

$$\begin{aligned} (f \star_M g)(x) &= f(x) \exp \left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu \right) g(x) \\ &= f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2} \right)^n \frac{1}{n!} \theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n} \partial_{\mu_1} \dots \partial_{\mu_n} f(x) \partial_{\nu_1} \dots \partial_{\nu_n} g(x) \end{aligned} \quad (2)$$

and

$$(f \star_V g)(x) = f(x) \exp \left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu + \frac{\vartheta}{2} \overleftarrow{\partial}_\mu \delta^{\mu\nu} \overrightarrow{\partial}_\nu \right) g(x). \quad (3)$$

For simplicity in the second case, we assume that the matrix $\theta^{\mu\nu}$ has a canonical block-diagonal form with the same parameter $\vartheta > 0$ in every 2×2 block $\begin{pmatrix} 0 & \vartheta \\ -\vartheta & 0 \end{pmatrix}$, although different blocks may have different values of ϑ , including zero in applications. These two star products play a central role in the Weyl-Wigner approach to quantum mechanics because the Moyal product is compatible with the symplectic structure of the linear phase space and both of them are compatible with its complex structure. It is easy to see that these two products yield the same commutation relation (1) for the coordinate functions. There is a simple relation between these products:

$$T(f \star_M g) = T(f) \star_V T(g), \quad \text{where} \quad T = e^{\frac{\vartheta}{4}\Delta}, \quad \Delta = \sum_{\mu=1}^d \partial_\mu^2. \quad (4)$$

This relation holds in the sense of formal power series and was discussed, in particular, in [15] in the quantum mechanics context. But from the standpoint of strict deformation quantization, products (2) and (3) and the operator T are well defined only under rather severe restrictions on the functions. In the quantum field theory formalism, the standard practice is to use the Schwartz space $S(\mathbb{R}^d)$ of rapidly decreasing smooth functions and it is commonly assumed that this space is an algebra under the Moyal multiplication. But we note that series (2) generally diverges for functions in $S(\mathbb{R}^d)$. In fact, using the classical Borel theorem (see, e.g., [16]) we can construct functions $f, g \in S(\mathbb{R}^d)$ such that the sequence $f(\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu)^n g|_{x=x_0}$, $n = 0, 1, \dots$, coincides with any given number sequence. Moreover, it is easily shown [13] that the series defining $f \star_M f$ does not converge in the topology of $S(\mathbb{R}^d)$ even for the Gaussian function $f(x) = e^{-\gamma|x|^2}$ if $\gamma > 1/\vartheta$. Nevertheless the multiplication \star_M is well defined on an analytic function space that we describe in Sec. VI. This space is dense in $S(\mathbb{R}^d)$ and the multiplication \star_M has a unique continuous extension to the Schwartz space; this extension can be written in terms of the Fourier transforms as

$$(f \star_M g)(x) = \frac{1}{(2\pi)^{2d}} \iint \hat{f}(k) \hat{g}(k') e^{i(k+k') \cdot x - \frac{i}{2} k \theta k'} dk dk', \quad (5)$$

where $k \theta k' \stackrel{\text{def}}{=} k_\mu \theta^{\mu\nu} k'_\nu$ is a symplectic inner product on \mathbb{R}^d .

It is easily seen that $S(\mathbb{R}^d)$ is an associative topological algebra with respect to the product defined by (5). The operator $T = e^{\vartheta\Delta/4}$ also has a natural extension to $S(\mathbb{R}^d)$ coinciding with multiplication of the Fourier transforms by $e^{-\vartheta|k|^2/4}$. Its image $\text{im } T$ consists of all entire functions satisfying the inequalities

$$|f(x + iy)| \leq C_N(1 + |x|)^{-N} e^{|y|^2/\vartheta}, \quad N = 0, 1, \dots$$

Such functions form an algebra under the Wick-Voros product or, more precisely, under the multiplication

$$(f \star_V g)(x) = \frac{1}{(2\pi)^{2d}} \iint \hat{f}(k) \hat{g}(k') e^{i(k+k') \cdot x - \frac{i}{2} k \theta k' - \frac{\vartheta}{2} k k'} dk dk', \quad (6)$$

where $kk' = \sum_{\mu} k_{\mu} k'_{\mu}$ is the Euclidean scalar product on \mathbb{R}^d . But the whole space $S(\mathbb{R}^d)$ cannot be made into a \star_V -algebra.

Proposition. *The Wick-Voros star product does not admit a continuous extension to the Schwartz space.*

Proof. Let u be the linear functional $f \rightarrow \int f(x) dx = \hat{f}(0)$. Clearly, $u \in S'$. For any $f \in \text{im } T$, we have

$$u(f \star_V f) = \frac{1}{(2\pi)^d} \int \hat{f}(k) \hat{f}(-k) e^{\frac{\vartheta}{2}|k|^2} dk.$$

We set $\hat{f}(k) = e^{-\vartheta|k|^2/4}$ and consider the sequence $\hat{f}_n(k) = e^{-|k|^4/n} \hat{f}(k)$ which obviously belongs to $\text{im } T$ and converges to \hat{f} in the Fourier-invariant space $S(\mathbb{R}^d)$. We suppose that there is a continuous extension of product (6) to this space. Then there would exist a function $g \in S(\mathbb{R}^d)$ such that $g = f \star_V f$ and $u(f_n \star_V f_n) \rightarrow \hat{g}(0)$. But

$$u(f_n \star_V f_n) = \frac{1}{(2\pi)^d} \int e^{-\frac{2}{n}|k|^4} dk \rightarrow \infty,$$

and this contradiction completes the proof.

Similar remarks can be made about the twisted tensor product. In the Moyal case it is defined by

$$(f_1 \otimes_M \cdots \otimes_M f_n)(x_1, \dots, x_n) = \prod_{1 \leq a < b \leq n} e^{\frac{i}{2} \partial_{x_a} \theta \partial_{x_b}} f_1(x_1) \cdots f_n(x_n), \quad (7)$$

where

$$\partial_{x_a} \theta \partial_{x_b} = \frac{\partial}{\partial x_a^{\mu}} \theta^{\mu\nu} \frac{\partial}{\partial x_b^{\nu}},$$

and is often called the “star product at different points” The star product itself is obtained from the twisted tensor product by identifying these points,

$$(f_1 \star_M \cdots \star_M f_n)(x) = (f_1 \otimes_M \cdots \otimes_M f_n)|_{x_1 = \cdots = x_n = x},$$

in complete analogy with the case of the ordinary pointwise product, which is obtained from the ordinary tensor product of functions by the same identification. In the Wick-Voros case we have

$$(f_1 \otimes_V \cdots \otimes_V f_n)(x_1, \dots, x_n) = \prod_{1 \leq a < b \leq n} e^{\frac{i}{2} \partial_{x_a} \theta \partial_{x_b} + \frac{\vartheta}{2} \partial_a \partial_b} f_1(x_1) \cdots f_n(x_n), \quad (8)$$

where

$$\partial_a \partial_b = \sum_{\mu} \frac{\partial}{\partial x_a^{\mu}} \frac{\partial}{\partial x_b^{\mu}}.$$

The twisted tensor product is associative and can be generalized to functions of several variables:

$$(f \otimes_M g)(x_1, \dots, x_m; x'_1, \dots, x'_n) = \prod_{a=1}^m \prod_{b=1}^n e^{\frac{i}{2} \partial_{x_a} \theta \partial_{x'_b}} f(x_1, \dots, x_m) g(x'_1, \dots, x'_n), \quad (9)$$

where the right-hand side is determined by the requirement of compatibility with (7) for $f = f_1 \otimes_M \dots \otimes_M f_m$ and $g = g_1 \otimes_M \dots \otimes_M g_n$. The product \otimes_M is well defined on sufficiently smooth functions and extends continuously to the Schwartz space. In terms of the Fourier transforms, the relation between the Moyal tensor product and the ordinary one takes the simple form

$$(\widehat{f \otimes_M g})(k, k') = e^{-\frac{i}{2} k \theta k'} (\hat{f} \otimes \hat{g})(k, k'), \quad f, g \in S(\mathbb{R}^d).$$

The exponential factor here is a multiplier for $S(\mathbb{R}^{2d})$ and hence defines an automorphism of this space. An analogous relation holds for $f_1 \otimes_M \dots \otimes_M f_n$ with the multiplier

$$\mu_n(k) = \prod_{1 \leq a < b \leq n} e^{-\frac{i}{2} k_a \theta k_b}. \quad (10)$$

This together with the Schwartz kernel theorem implies that for every continuous multilinear form

$$v: \underbrace{S(\mathbb{R}^d) \times \dots \times S(\mathbb{R}^d)}_n \rightarrow \mathbb{C},$$

there is a unique linear functional $w \in S'(\mathbb{R}^{nd})$ such that the diagram

$$\begin{array}{ccc} S(\mathbb{R}^d) \times \dots \times S(\mathbb{R}^d) & \xrightarrow{v} & \mathbb{C} \\ \otimes_M \downarrow & \nearrow w & \\ S(\mathbb{R}^{nd}) & & \end{array} \quad (11)$$

is commutative.

III. Twist-deformed quantum fields

We now turn to the noncommutative deformation [8, 9] of quantum field theories that can be associated with the Moyal tensor product. This construction does not use the Lagrangian formulation and is model independent, but we here restrict our consideration to the case of one scalar field $\phi(x)$ for simplicity. Our starting point is a quantum field theory on commutative Minkowski space with the usual assumptions [17, 18] of relativistic covariance, locality, and physical stability expressed by the spectral condition. We also make the standard assumptions about the domain and continuity of the field. Then every vacuum expectation value is well defined as a tempered distribution

$$\langle \Psi_0, \phi(f_1) \dots \phi(f_n) \Psi_0 \rangle = (w^{(n)}, f_1 \otimes \dots \otimes f_n), \quad w^{(n)} \in S'(\mathbb{R}^{4n}). \quad (12)$$

The idea is to deform these distributions by replacing the tensor product of their arguments with the twisted tensor product

$$(w_\theta^{(n)}, f_1 \otimes \cdots \otimes f_n) \stackrel{\text{def}}{=} (w^{(n)}, f_1 \otimes_M \cdots \otimes_M f_n). \quad (13)$$

It follows from what has been said above that there exists a unique $w_\theta^{(n)} \in S'(\mathbb{R}^{4n})$ satisfying (13) and by the Wightman reconstruction theorem, the sequence of deformed distributions $w_\theta^{(n)}$ can be used to construct a field theory. It is notable that there is in fact no need to appeal to the reconstruction theorem because we can easily directly define a quantum field ϕ_θ whose vacuum expectation values coincide with the deformed Wightman functions $w_\theta^{(n)}$. Namely, the Schwartz kernel theorem gives a precise meaning to all vectors of the form

$$\Phi_n(f) = \int dx_1 \dots dx_n \phi(x_1) \cdots \phi(x_n) f(x_1, \dots, x_n) \Psi_0, \quad (14)$$

where Ψ_0 is the vacuum state and f ranges the space $S(\mathbb{R}^{4n})$. Therefore, we may take the linear span D of all such vectors as a domain of the initial field ϕ . For each $g \in S(\mathbb{R}^4)$, we define $\phi_\theta(g)$ by

$$\phi_\theta(g) \Psi_0 = \phi(g) \Psi_0, \quad \phi_\theta(g) \Phi_n(f) = \Phi_{n+1}(g \otimes_M f), \quad n \geq 1, \quad (15)$$

extended by linearity. The properties of the fields ϕ_θ can be summarized as follows [8].

Theorem 1. *Let ϕ be a Hermitian scalar field satisfying the Wightman axioms and let $w^{(n)}$ be its Wightman functions. Then the deformed fields ϕ_θ are well defined as operator-valued tempered distributions with the same domain in the Hilbert space of ϕ , and*

$$\langle \Psi_0, \phi_\theta(g_1) \cdots \phi_\theta(g_n) \Psi_0 \rangle = (w^{(n)}, g_1 \otimes_M \cdots \otimes_M g_n), \quad g_j \in S(\mathbb{R}^4). \quad (16)$$

The vacuum state Ψ_0 of ϕ is a cyclic vector for every field ϕ_θ . These fields satisfy the hermiticity condition

$$\phi_\theta(g)^* \supset \phi_\theta(\bar{g}), \quad g \in S(\mathbb{R}^4), \quad (17)$$

and their vacuum expectation values $w_\theta^{(n)}$ satisfy the spectral condition as well as the positive definiteness condition

$$\sum_{m,n=0}^N (w_\theta^{(m+n)}, f_m^* \otimes f_n) \geq 0, \quad (18)$$

where $w_\theta^{(0)} = 1$, $f_0 \in \mathbb{C}$, $f^*(x_1, \dots, x_m) \stackrel{\text{def}}{=} \overline{f(x_m, \dots, x_1)}$, and $\{f_j\}_{1 \leq j \leq N}$ is an arbitrary finite set of test functions such that $f_j \in S(\mathbb{R}^{4j})$.

Proof. For any $g \in S(\mathbb{R}^4)$, $f \in S(\mathbb{R}^{4n})$, and $h \in S(\mathbb{R}^{4m})$, we have

$$\langle \Phi_m(h), \Phi_{n+1}(g \otimes_M f) \rangle = \langle \Phi_{m+1}(\bar{g} \otimes_M h), \Phi_n(f) \rangle, \quad (19)$$

or, equivalently,

$$(w^{(m+n+1)}, h^* \otimes (g \otimes_M f)) = (w^{(m+n+1)}, (\bar{g} \otimes_M h)^* \otimes f). \quad (20)$$

Indeed, it follows from definition (9) that $(\bar{g} \otimes_M h)^* = h^* \otimes_M g$ because the matrix θ is antisymmetric. Furthermore,

$$\begin{aligned} \widehat{h^* \otimes (g \otimes_M f)} &= (\hat{h}^* \otimes \hat{g} \otimes \hat{f}) e^{-\frac{i}{2} k \theta (\sum_{a=1}^n p_a)}, \\ \widehat{(h^* \otimes_M g) \otimes f} &= (\hat{h}^* \otimes \hat{g} \otimes \hat{f}) e^{-\frac{i}{2} (\sum_{b=1}^m q_b) \theta k}, \end{aligned}$$

where k , p_a , and q_b are the respective arguments of the functions \hat{g} , \hat{f} , and \hat{h}^* . Again using the antisymmetry of θ , we obtain relation (20) because $\hat{w}^{(m+n+1)}(q_1, \dots, q_m, k, p_1, \dots, p_n)$ contains the factor $\delta(k + \sum_{b=1}^m q_b + \sum_{a=1}^n p_a)$ expressing the translation invariance of $w^{(m+n+1)}$. It follows from (19) that if a linear combination Φ of vectors of form (14) is zero, then $\phi_\theta(g)\Phi$ is also zero, i.e., the operators $\phi_\theta(g)$ are well defined. It is also obvious that $\phi_\theta(g)\Phi \rightarrow 0$ as $g \rightarrow 0$ in $S(\mathbb{R}^4)$. Moreover, (19) implies (17).

Identity (16) follows directly from (15). If a vector Ψ is such that $\langle \Psi, \phi_\theta(g_1) \cdots \phi_\theta(g_n) \Psi_0 \rangle = 0$ for arbitrary $g_1, \dots, g_n \in S(\mathbb{R}^4)$ and each n , then $\langle \Psi, \Phi_n(f) \rangle = 0$ for every $\Phi_n(f)$ of form (14) because $\langle \Psi, \Phi_n(f) \rangle$ is just the element of $S'(\mathbb{R}^{4n})$ that is associated with the multilinear form $(g_1, \dots, g_n) \rightarrow \langle \Psi, \phi_\theta(g_1) \cdots \phi_\theta(g_n) \Psi_0 \rangle$ by diagram (11). Therefore, $\Psi = 0$, and the linear span of vectors $\phi_\theta(g_1) \cdots \phi_\theta(g_n) \Psi_0$ is hence dense in the Hilbert space. Clearly, $\hat{w}_\theta^{(n)} = \mu_n \cdot \hat{w}^{(n)}$, where μ_n is given by (10), and the deformed Wightman functions therefore have the same spectral properties as those of $w^{(n)}$'s. To prove (18), we note that $(w_\theta^{(m+n)}, f_m^* \otimes f_n) = (w^{(m+n)}, g_m^* \otimes_M g_n)$, where $\hat{g}_m = \mu_m \cdot \hat{f}_m$ and $\hat{g}_n = \mu_n \cdot \hat{f}_n$. Furthermore, by the antisymmetry of θ and the translation invariance of $w^{(m+n)}$, we have $(w^{(m+n)}, h \otimes_M g) = (w^{(m+n)}, h \otimes g)$ for each $h \in S(\mathbb{R}^{dm})$ and $g \in S(\mathbb{R}^{dn})$ because $\widehat{h \otimes_M g} = (\hat{h} \otimes \hat{g})e^{-\frac{i}{2}(\sum_1^m q_b)\theta(\sum_1^n p_a)}$. The theorem is proved.

By the standard argument [17], any other implementation of field theory with vacuum expectation values (16) and with a cyclic vacuum state invariant under translations is unitary equivalent to this one. If the initial field ϕ is free, then its creation and annihilation operators are deformed as follows

$$a_\theta(p) = e^{\frac{i}{2}p\theta P} a(p), \quad a_\theta^*(p) = e^{-\frac{i}{2}p\theta P} a^*(p),$$

where P is the energy-momentum operator. Accordingly, they satisfy the deformed canonical commutation relations (CCRs)

$$\begin{aligned} a_\theta(p)a_\theta(p') &= e^{-ip\theta p'} a_\theta(p')a_\theta(p), \quad a_\theta^*(p)a_\theta^*(p') = e^{-ip\theta p'} a_\theta^*(p')a_\theta^*(p), \\ a_\theta(p)a_\theta^*(p') &= e^{ip\theta p'} a_\theta^*(p')a_\theta(p) + 2\omega_{\mathbf{p}}\delta(\mathbf{p} - \mathbf{p}'). \end{aligned} \quad (21)$$

Twisted CCR algebra (21) was discussed from different standpoints in [6, 19, 20, 21, 22]. As already stated, this deformation preserves the translation invariance, but it obviously violates the Lorentz invariance. The full transformation law of the deformed fields under the proper Poincaré group is given by

$$U(a, \Lambda)\phi_\theta(f)U^{-1}(a, \Lambda) = \phi_{\Lambda\theta\Lambda^T}(f_{a,\Lambda}), \quad (22)$$

where $f_{(a,\Lambda)}(x) = f(\Lambda^{-1}(x - a))$, $(a, \Lambda) \in \mathcal{P}_+^\uparrow$. Hence, every ϕ_θ transforms covariantly only under those Lorentz transformations that leave the matrix $\theta^{\mu\nu}$ unchanged.

This deformation also leads to the lack of locality and the fields ϕ_θ do not satisfy the standard microcausality condition. This can be easily seen by considering the matrix elements

$$M_\Phi(x, x') = \langle \Psi_0, [\phi_\theta(x), \phi_\theta(x')] \Phi \rangle \quad (23)$$

in the simplest case of a free field. If we take Φ to be a normalized two-particle state

$$\Phi = \phi^{(-)}(h)\phi^{(-)}(h)\Psi_0, \quad \text{where } h \in S(\mathbb{R}^d), \quad (24)$$

then a simple direct calculation shows that the matrix element (23) is nonzero for some spacelike separated points. Moreover, as shown in [8], this distribution does not vanish in any open region, i.e., $\text{supp } M_\Phi = \mathbb{R}^{2,4}$. The reason is that its Fourier transform has the form

$$\widehat{M}_\Phi(k, k') = -4i\hat{w}(k)\hat{w}(k')\hat{h}(k)\hat{h}(k')\sin\frac{k\theta k'}{2},$$

where $\hat{w}(k)$ is the momentum-space two-point function of ϕ . Hence \widehat{M}_Φ has support in the product $\bar{V}_+ \times \bar{V}_+$ of two closed forward cones. The factor $(\hat{h} \otimes \hat{h})\sin\frac{k\theta k'}{2}$ does not vanish on $\text{supp}(\hat{w} \otimes \hat{w})$ if $\Phi \neq 0$. Therefore, M_Φ is the boundary value of a nonzero analytic function. Such a distribution cannot vanish on any open nonempty set by the general uniqueness theorem for analytic functions (see Theorem B.7 in [18]). We can therefore conclude that the noncommutative deformation by twisting products in the vacuum expectation values brings the causality principle and the spectral condition into conflict.

IV. Localization in wedge-shaped regions

The deformed fields ϕ_θ with different θ retain some relative localization properties, established in [9, 21], where it was noted that a wedge-shaped space-time region W_θ can be associated with each antisymmetric matrix θ . If this matrix has the standard form

$$\theta_1 = \begin{pmatrix} 0 & \vartheta_e & 0 & 0 \\ -\vartheta_e & 0 & 0 & 0 \\ 0 & 0 & 0 & \vartheta_m \\ 0 & 0 & -\vartheta_m & 0 \end{pmatrix}, \quad \vartheta_e \geq 0,$$

then its associated wedge W_1 is defined by the inequality $x^1 > |x^0|$. The stabilizer subgroup of θ_1 with respect to the action $\theta \rightarrow \Lambda\theta\Lambda^T$ of the proper Lorentz group \mathcal{L}_+ coincides with that of W_1 with respect to the action $W \rightarrow \Lambda W$, and there is therefore a one-to-one correspondence between the orbits of θ_1 and W_1 . Grosse and Lechner [9, 21] have shown that the fields ϕ_θ satisfy the following wedge-local commutation relation: if the sets $\text{supp } f + W_\theta$ and $\text{supp } g + W_{\theta'}$ are spacelike separated, then

$$[\phi_\theta(f), \phi_{\theta'}(g)]\Psi = 0 \quad \text{for all } \Psi \in D.$$

Therefore, the deformed fields should be regarded as objects localizable in wedge-shaped regions and not at points of space-time.

Such a localization is similar to that studied in [23] in the framework of algebraic quantum field theory. As noted in [9, 21], even this weak form of local commutativity allows constructing a scattering theory. But this type of localization is radically different from the replacement of the light cone with a light wedge proposed previously as a possible modification of the microcausality condition in field theories with space-space noncommutativity corresponding to the low-energy limit of string theory. In the next section, we discuss an instructive example giving an idea of such a modification.

v. Twist-deformed Wick square of a free field

We consider the normal ordered square of a free scalar field in the four-dimensional space-time but change the ordinary product in its definition to the Moyal star product:

$$\begin{aligned} \mathcal{O}(x) \stackrel{\text{def}}{=} \phi \star_M \phi : (x) = \lim_{x_1, x_2 \rightarrow x} : \phi(x_1) \phi(x_2) : \\ + \sum_{n=1}^{\infty} \left(\frac{i}{2} \right)^n \frac{1}{n!} \theta^{\mu_1 \nu_1} \dots \theta^{\mu_n \nu_n} \lim_{x_1, x_2 \rightarrow x} : \partial_{\mu_1} \dots \partial_{\mu_n} \phi(x_1) \partial_{\nu_1} \dots \partial_{\nu_n} \phi(x_2) : \dots \end{aligned} \quad (25)$$

It is easy to see that in the case of space-space noncommutativity, where $\theta^{23} = -\theta^{32} = \vartheta \neq 0$ and the other elements of the matrix θ are zero, the commutator $[\mathcal{O}(x), \mathcal{O}(y)]$ vanishes in the wedge defined by the inequality $|x^0 - y^0| < |x^1 - y^1|$. It was shown in [24] that an analogous commutator with the time derivative is nonzero at some points outside this wedge:

$$\langle 0 | [\mathcal{O}(x), \partial_0 \mathcal{O}(y)]_- | p_1, p_2 \rangle |_{x^0=y^0} \neq 0.$$

It was conjectured in that paper that this result holds generally when there are time derivatives in the observables. A closer examination in [8] showed that the space-space noncommutativity violates the microcausality condition even if there are no time derivatives in the observables. Moreover, the commutator $[\mathcal{O}(x), \mathcal{O}(y)]$ is nonzero for any points \bar{x} and \bar{y} outside the wedge $|x^0 - y^0| < |x^1 - y^1|$. In more precise terms, there is a state Φ such that (\bar{x}, \bar{y}) belongs to the support of the distribution $\langle \Psi_0, [\mathcal{O}(x), \mathcal{O}(y)] \Phi \rangle$.

The star commutator

$$[\mathcal{O}(x), \mathcal{O}(y)]_\star = \mathcal{O}(x) \star_M \mathcal{O}(y) - \mathcal{O}(y) \star_M \mathcal{O}(x)$$

was also discussed in the literature with contradictory conclusions. An analysis of its matrix elements in the same paper [8] showed that it is nonzero everywhere. In particular, these results demonstrate that the power series expansions of the matrix elements in the noncommutativity parameter do not converge in the topology of the space of tempered distributions, because their partial sums obviously satisfy the microcausality condition.

This simple example also demonstrates that noncommutative field theory with the Wick-Voros star product is more nonlocal than that with the Moyal product. A calculation analogous to that in [8] shows that in the Wick-Voros case, the matrix element of the corresponding commutator between the vacuum and two-particle state (24) has the form

$$\begin{aligned} \langle \Psi_0, [\mathcal{O}_V(x), \mathcal{O}_V(y)] \Phi \rangle = 8 \iiint \epsilon(k^0) \delta(k^2 - m^2) e^{\frac{\vartheta}{2} \mathbf{k}(\mathbf{p}_2 - \mathbf{p}_1)} e^{-ik \cdot (x-y) - ip_1 \cdot x - ip_2 \cdot y} \\ \times \prod_{a=1}^2 \theta(p_a^0) \delta(p_a^2 - m^2) \cos \left(\frac{1}{2} k \theta p_a \right) \hat{h}(p_a) \frac{dk dp_1 dp_2}{(2\pi)^{12}}, \end{aligned}$$

where $\mathbf{k} = (k^2, k^3)$. This expression contains an additional exponential factor $e^{\frac{\vartheta}{2} \mathbf{k}(\mathbf{p}_2 - \mathbf{p}_1)}$ compared with the Moyal case and is therefore not a tempered distribution and is well defined in coordinate space only on analytic test functions.

VI. Convergence of star products and adequate spaces of functions

We now turn to the conditions for the convergence of the star products and to the description of the test function spaces [13, 14] that are most suitable for a noncommutative quantum field theory. As stated above, the formal power series representing the Moyal and Wick-Voros star products generally diverge for functions belonging to the Schwartz space. But both of them converge under the additional condition

$$(1 + |x|)^N |\partial^\kappa f(x)| < C_N B^{|\kappa|} \sqrt{\kappa!}, \quad (26)$$

where C_N and B are constants depending on f and

$$B < \frac{1}{\sqrt{|\theta|}}, \quad |\theta| = \max |\theta^{\mu\nu}|. \quad (27)$$

In (26), the multi-index notation $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{Z}_+^d$, $\partial^\kappa = \partial_{x_1}^{\kappa_1} \dots \partial_{x_d}^{\kappa_d}$, $|\kappa| = \kappa_1 + \dots + \kappa_d$, and $\kappa! = \kappa_1! \dots \kappa_d!$ is used. The convergence of the Moyal product under the indicated condition was proved in two different ways in [13, 14]. But the function space defined by (26) is not an algebra even with respect to the usual pointwise multiplication. To obtain an algebra, we take those functions that satisfy analogous inequalities with an arbitrary small B and let $\mathcal{S}^{1/2}$ denote this space. It has a natural topology determined by the set of norms

$$\|f\|_{B,N} = \sup_{x,\kappa} (1 + |x|)^N \frac{|\partial^\kappa f(x)|}{B^{|\kappa|} \sqrt{\kappa!}}.$$

It is easy to see that $\mathcal{S}^{1/2}$ is a Fréchet space, i.e., is metrizable and complete. Moreover, it is a nuclear space, as follows from a result in [25], where it was established that the Gelfand-Shilov spaces S^β are nuclear. But we emphasize that $\mathcal{S}^{1/2} \neq S^{1/2}$. The space $S^{1/2}$ is the inductive limit of the family of spaces $S^{1/2,B}$ as $B \rightarrow \infty$ (and, in particular, is nonmetrizable), while $\mathcal{S}^{1/2}$ is the projective limit of the same family as $B \rightarrow 0$. The spaces S^β with $\beta < 1/2$ were proposed for using in noncommutative QFT in [26]. The nuclearity is crucial in deriving the Schwartz kernel theorem, which plays an important role in axiomatic formulation of quantum field theory. A simple proof of an analogous kernel theorem for a large class of spaces including $\mathcal{S}^{1/2}$ and S^β was given in [27, 28]. The following result extends Theorem 6 in [14].

Theorem 2. *The space $\mathcal{S}^{1/2}(\mathbb{R}^d)$ is a topological algebra with respect to both the Moyal star product and the Wick-Voros star product. If $f, g \in \mathcal{S}^{1/2}(\mathbb{R}^d)$, then the series representing these products converge absolutely in this space. Moreover these products depend continuously on the noncommutativity parameter θ , and the operator $T = e^{\frac{\theta}{4}\Delta}$ is an automorphism of this space.*

Proof. We first describe the infinite-order differential operators

$$\sum_{\lambda \in \mathbb{Z}_+^d} c_\lambda \partial^\lambda \quad (28)$$

that act continuously on $\mathcal{S}^{1/2}$. We assume that $\sum_\lambda c_\lambda z^\lambda$ has less than exponential growth of order 2 and type b , which is equivalent to the condition

$$|c_\lambda| \leq C(2b)^{|\lambda|/2} \frac{1}{\sqrt{\lambda!}}. \quad (29)$$

Applying (28) to $f \in \mathcal{S}^{1/2}$, we obtain

$$\begin{aligned} (1 + |x|)^N \left| \partial^\kappa \sum_\lambda c_\lambda \partial^\lambda f(x) \right| &\leq \|f\|_{B,N} \sum_\lambda c_\lambda B^{|\kappa+\lambda|} \sqrt{(\kappa+\lambda)!} \\ &\leq \|f\|_{B,N} 2^{|\kappa|/2} B^{|\kappa|} \sqrt{\kappa!} \sum_\lambda c_\lambda 2^{|\lambda|/2} B^{|\lambda|} \sqrt{\lambda!}. \end{aligned} \quad (30)$$

We also assume that

$$b < \frac{1}{4B^2}. \quad (31)$$

Then the last series in (30) converges, and taking $B' \geq B\sqrt{2}$, we obtain

$$\left\| \sum_\lambda c_\lambda \partial^\lambda f \right\|_{B',N} \leq C' \|f\|_{B,N}.$$

Therefore, operator (28) with any $b < \infty$ maps $\mathcal{S}^{1/2}$ into itself continuously. We note that the operator in definition (2) of the \star_M -product has a type of growth that is obviously no greater than $|\theta|/4$ and (27) hence implies (31). If the matrix θ has the standard form indicated in Sec. II, then b is simply ϑ . We conclude that $T = e^{\vartheta\Delta/4}$ is an automorphism of $\mathcal{S}^{1/2}(\mathbb{R}^d)$.

Further, the operators in definitions (7) and (8) are automorphisms of $\mathcal{S}^{1/2}(\mathbb{R}^{nd})$, and the series representing the twisted tensor products converge absolutely in $\mathcal{S}^{1/2}(\mathbb{R}^{nd})$. Using the Leibnitz formula, we can easily verify that $\mathcal{S}^{1/2}$ is a topological algebra under the pointwise multiplication. It follows that the map $\mathcal{S}^{1/2}(\mathbb{R}^{2d}) \rightarrow \mathcal{S}^{1/2}(\mathbb{R}^d): f(x, y) \rightarrow f(x, x)$ is continuous because this map is just the linear map associated with the pointwise multiplication and $\mathcal{S}^{1/2}(\mathbb{R}^{2d}) = \mathcal{S}^{1/2}(\mathbb{R}^d) \hat{\otimes} \mathcal{S}^{1/2}(\mathbb{R}^d)$ by the kernel theorem. The Moyal and Wick-Voros star products (2) and (3) are obtained from $f \otimes_M g$ and $f \otimes_V g$ by this map and are therefore also continuous. Their continuity in θ can be established in the same way as in the proof of Theorem 5 in [13]. The theorem is proved.

It is essential that $\mathcal{S}^{1/2}$ is the largest subspaces of the Schwartz space S with the properties established in Theorem 2. But using only $\mathcal{S}^{1/2}$ does not suffice to describe the nonlocal features introduced by noncommutativity exactly. To better understand the causal structure of noncommutative models, it is useful to examine the decay properties of the field commutators more closely. A simple, well-known way to describe the decay properties of a distribution is to consider its convolution with sufficiently rapidly decreasing test functions. But the functions satisfying the convergence condition stated above cannot decrease too rapidly. The most suitable space consists of functions with a Gaussian decrease. More precisely, it is reasonable to use a two-parameter family of spaces $S_{1/2,A}^{1/2,B}$, each consisting of functions with the finite norm

$$\|f\|_{A,B} = \sup_{\kappa,x} e^{|x/A|^2} \frac{|\partial^\kappa f(x)|}{B^{|\kappa|} \sqrt{\kappa!}}$$

As shown in the appendix in [8], this space is nontrivial if $AB > 2$ and becomes trivial if $AB < \sqrt{2}$.

VII. θ -locality

Using rapidly decreasing test functions, we can show that certain causality properties hold independently of the type of noncommutativity. For this, we again consider the deformed

normal ordered square $\mathcal{O}(x) =: \phi \star \phi : (x)$ of a free field but average it with such functions this time,

$$\mathcal{O}(f_a) = \int dx \mathcal{O}(x) f(x-a), \quad f \in S_{1/2,A}^{1/2,B}.$$

We must estimate the asymptotic behavior of the commutator $[\mathcal{O}(f_a), \mathcal{O}(g_{-a})]$ of two averaged observables at large spacelike separations. We fix a spacelike vector a and let γ denote the angular distance of this vector from the light cone,

$$\gamma = \inf_{\xi^2 \geq 0} |\xi - a/|a||.$$

Evaluating the matrix element

$$\langle \Psi_0, [\mathcal{O}(f_a), \mathcal{O}(g_{-a})] \Phi \rangle \quad (32)$$

between the vacuum and a two-particle state $\Phi = \phi^{(-)}(h_1)\phi^{(-)}(h_2)\Psi_0$, we find [8] that it decreases in a Gaussian manner as $|a|$ increases:

$$|\langle \Psi_0, [\mathcal{O}(f_a), \mathcal{O}(g_{-a})] \Phi \rangle| \leq C_{\Phi,A'} \|f\|_{A,B} \|g\|_{A,B} e^{-2|\gamma a/A'|^2} \quad (33)$$

for all $A' > A$. It follows from the convergence condition $B < 1/\sqrt{|\theta|}$ and the nontriviality condition $AB > 2$, that the best result is at $A \sim \sqrt{|\theta|}$. We therefore conclude that matrix element (32) decreases like $e^{-|\gamma a|^2/|\theta|}$ as the spacelike separation tends to infinity. This result holds for both the Moyal and the Wick-Voros products. Moreover, it can be shown that any matrix element of the commutator behaves similarly.

Estimate (33) is quite informative, but it can be expressed in more abstract terms that are even more convenient for applications. This can be done using another class of function spaces which are associated with cone-shaped regions and defined as follows. Let U be a cone in \mathbb{R}^d . A smooth function f on \mathbb{R}^d belongs to the space $S^{1/2,B}(U)$ if it satisfies the condition

$$\sup_{x \in U} (1 + |x|)^N |\partial^\kappa f(x)| < C_N B^{|\kappa|} \sqrt{\kappa!}.$$

It turns out that the result (33) is equivalent to saying that for any $B < 1/\sqrt{|\theta|}$, the matrix element considered as a generalized function has a continuous extension to the space $S^{1/2,B}(\mathbb{V})$ associated with the relative light cone $\mathbb{V} = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : (x - y)^2 \geq 0\}$. This suggests how the microcausality condition could be generalized to quantum fields on noncommutative space-time.

The nonlocal effects in noncommutative field theory are determined by the structure of the star product, and we can expect that in this theory, all matrix elements $\langle \Phi, [\phi(x), \psi(x')]_- \Psi \rangle$ of field commutators (or anticomutators for unobservable fields) allow a continuous extension to such a space $S^{1/2,B}(\mathbb{V})$ whose superscript B is of the order $1/\sqrt{|\theta|}$ but in general may depend on the fields ϕ and ψ and the states Φ and Ψ . This condition has been introduced in [14] and we call it θ -locality for brevity. We emphasize that it is consistent with the Poincaré covariance. Conceivably, θ -locality expresses the absence of acausal effects on scales much larger than the fundamental length $\sqrt{|\theta|}$. If such is indeed the case, then this formulated condition might be called macrocausality. It is quite possible that the matrix elements of fields are tempered distributions in physically relevant noncommutative models. Even so, the space $S^{1/2,B}(\mathbb{V})$ can be considered as a tool for formulating causality, and it is clear from the above that its role can be more essential in the case of the Wick-Voros star product.

VIII. Conclusion

The physical consequences of the noncommutative quantum field theory obtained by twisting tensor products need further investigation. Here, we did not touch the twisted Poincaré covariance, but we note that naively coupling the deformation considered above with this covariance can lead to a theory physically equivalent to the undeformed one, analogously to the theory discussed in [20]. The issue of how best to combine the idea of space-time noncommutativity with the (properly adapted) basic principles of quantum physics still remains open. The θ -locality condition means that the commutators of observables behave at large spacelike separations like $\exp(-|x-y|^2/\theta)$, and this condition is similar to the asymptotic commutativity condition previously used in nonlocal QFT on the usual commutative Minkowski space. As shown in [29], the asymptotic commutativity in combination with the relativistic covariance and the spectral condition ensures the existence of the CPT symmetry and the usual spin-statistics relation for nonlocal fields. This is an additional argument for using the θ -locality to formulate causality in a noncommutative quantum field theory. The space $\mathcal{S}^{1/2}$, being a maximal topological star product algebra with absolute convergence, is completely adequate to the concept of strict deformation quantization and can be used to formulate and study noncommutative field models nonperturbatively.

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